XXII. Student Worksheet

We present four snapshots (ideas) contained in problems from the 2002 AMC 10/12 contests, one from each exam (10A, 10B, 12A, 12B), along with another problem based on each idea. Answers to the challenge problems can be found on page 2 of this manual: Changes and Important Procedures.

AMC 10A, Problem #16 – "Sum Them All"

- If $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$, then $a + b + c + d$ is

  (A) -5 (B) -10/3 (C) -7/3 (D) 5/3 (E) 5

- **Solution (B)** Because a, b, c, d appear exactly once in each of the left-hand side of equalities:

$$a + 1 = a + b + c + d + 5$$
$$b + 2 = a + b + c + d + 5$$
$$c + 3 = a + b + c + d + 5$$
$$d + 4 = a + b + c + d + 5$$

  sum up all these relations and get

$$(a + b + c + d) + 10 = 4(a + b + c + d) + 20.$$

  We obtain $a + b + c + d = -10/3$

You can use the same method to solve problem 19 on the 2002 AMC 10P:

- If a, b, c are real numbers such that $a^2 + 2b = 7$, $b^2 + 4c = -7$, and $c^2 + 6a = -14$, find $a^2 + b^2 + c^2$.

  (A) 14 (B) 21 (C) 28 (D) 35 (E) 49

AMC 10B, Problem #20 – "Use the Symmetry"

- Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

  (A) 0 (B) 1 (C) 4 (D) 7 (E) 8

- **Solution (B)** Exploit the symmetry with respect to the absolute values of the coefficients to a and c and with respect to the coefficient to b and the free term, respectively, by writing the given relations as $a + 8c = 4 + 7b$ and $8a - c = 7 - 4b$. Squaring both equations and adding up the results yields

$$(a + 8c)^2 + (8a - c)^2 = (4 + 7b)^2 + (7 - 4b)^2.$$

Expanding gives $65(a^2 + c^2) = 65(1 + b^2)$. So $a^2 + c^2 = 1 + b^2$, and $a^2 - b^2 + c^2 = 1$.

You can use the same technique to solve problem 27 on the 1999 AHSME:

- In triangle ABC, $3 \sin A + 4 \cos B = 6$ and $4 \sin B + 3 \cos A = 1$. Then $\angle C$ in degrees is:

  (A) 30 (B) 60 (C) 90 (D) 120 (E) 150